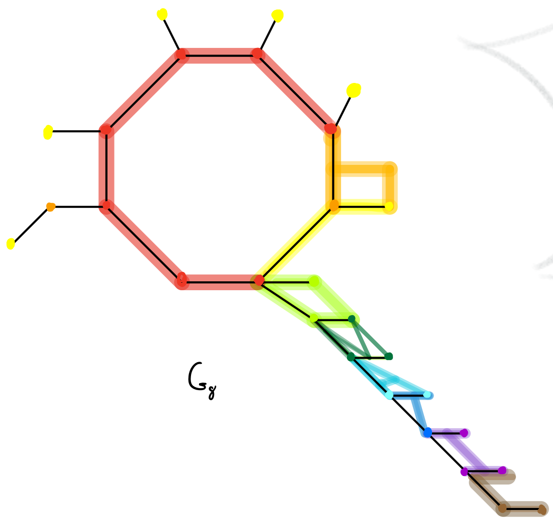
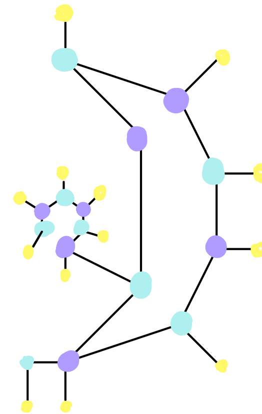


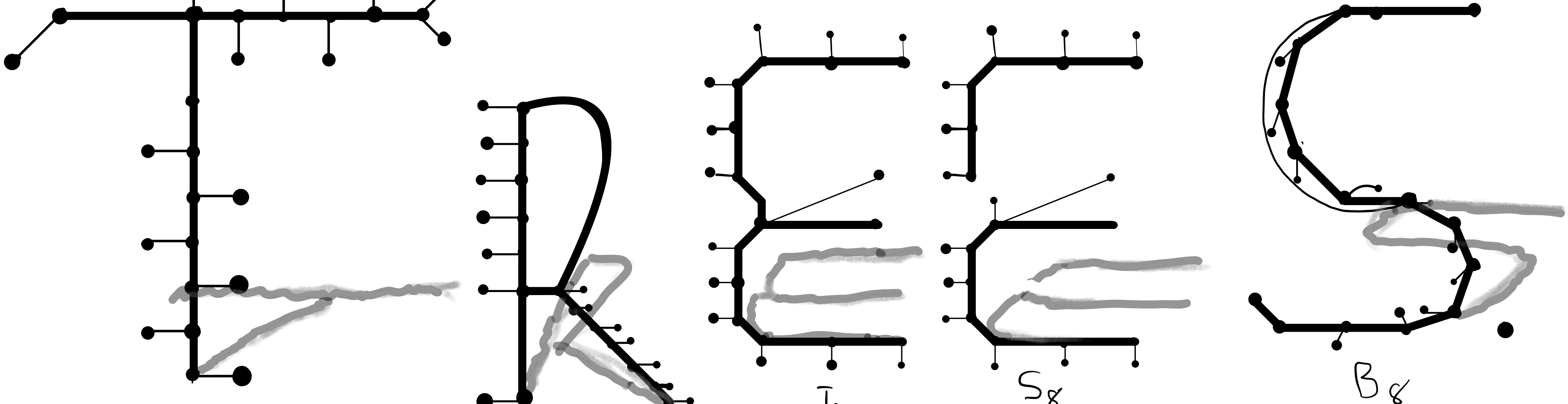
G_8

Light & Shade



H_8

ON THE



R_8 A_8 T_8 S_8 B_8

ARAM BINGHAM • UNIVERSIDAD DE CHILE • AlCoVE 2026

Plan/Game

(0) Thanks to the organizers and the...

INSTITUT TUTTE INSTITUTE

https://automathon.org/index.php/tutte-survey/

(1) Conjecture/Problem/Question

(2) Light

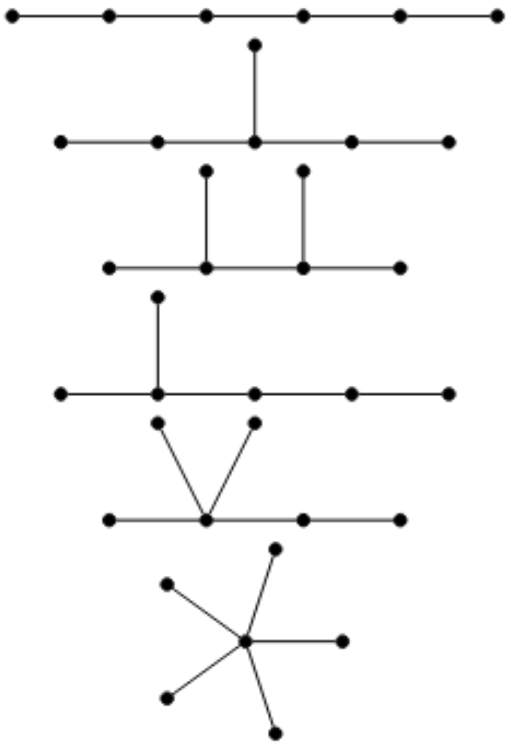
(3) Shade

The gang

BJLOPS: "The Chromatic Symmetric Function for Unicyclic Graphs", from an AMS MRC collaboration with Lisa Johnston, Colin Lawson, Rosa Orellana, Jianping Pan, and Chelsea Sato

A problem

Part 1: Chromatic symmetric functions and the tree isomorphism problem



Graph coloring and the chromatic polynomial

For a graph G with vertex set V, a **proper coloring** of G is a function $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(u) \neq \kappa(v)$ whenever u and v are adjacent.

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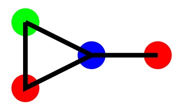

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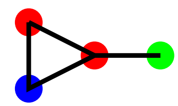

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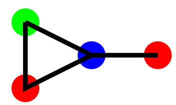

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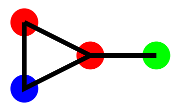

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Definition

A proper <u>k-coloring</u> uses only the colors $\{1, \ldots, k\}$. Let the number of k-colorings of a graph be denoted $\chi_G(k)$.

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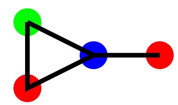

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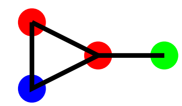

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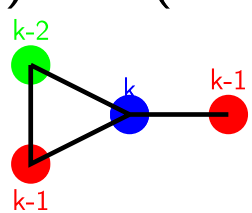

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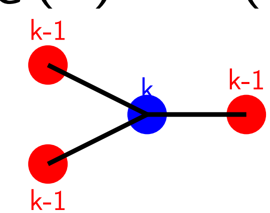

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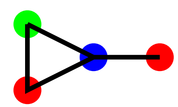

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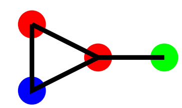

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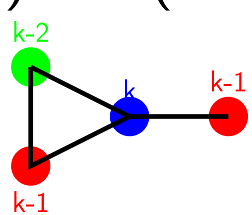

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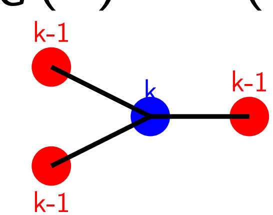

Fact: For any tree T on n vertices, $\chi_T(k) = k(k-1)^{n-1}$.

The CSF of a graph

Suppose $x_1, x_2, \ldots$ are commuting variables (think of these as colors).

Definition: (Stanley) The **chromatic symmetric function (CSF)** is

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to \mathbb{Z}_{>0}}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)} \,.$$

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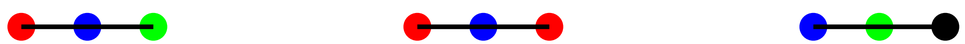

$x_1x_2x_3$ $\quad$ $x_1^2x_2$ $\quad$ $x_2x_3x_7$

Then $X_G = \ldots + x_1x_2x_3 + \ldots + x_1^2x_2 + \cdots + x_2x_3x_7 + \cdots$.

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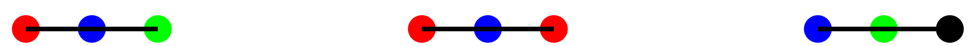

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Observe: The CSF is <u>homogeneous of degree n</u> (number of vertices in G) and specializes to the chromatic polynomial as

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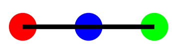

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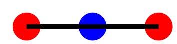

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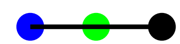

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Big Q: How much information about G can be recovered from X_G?

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Since X_G is symmetric and the set of all symmetric functions forms a vector space, X_G can be written as a linear combination in **any** basis.

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$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}.$$

EX: If $\lambda = (3, 2, 1)$, then

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Theorem: (Stanley, 1995) For a graph G with edge set E,

$$X_G = \sum_{A \subseteq E} (-1)^{|A|} p_{\lambda(A)}.$$

The tree isomorphism ~~conjecture~~ statement of ignorance

FIG. 1. Graphs G and H with $X_G = X_H$.

eleven nonisomorphic four-vertex graphs, but that there is a unique pair G, H of nonisomorphic five-vertex graphs for which $X_G = X_H$. These graphs are shown in Fig. 1. Using the notation of Proposition 2.4, we have

$$X_G = X_H = 2\tilde{m}_{221} + 4\tilde{m}_{2111} + \tilde{m}_{11111}.$$

We do not know whether X_G distinguishes trees. Of course all d-vertex trees have the same chromatic polynomial $n(n-1)^{d-1}$.

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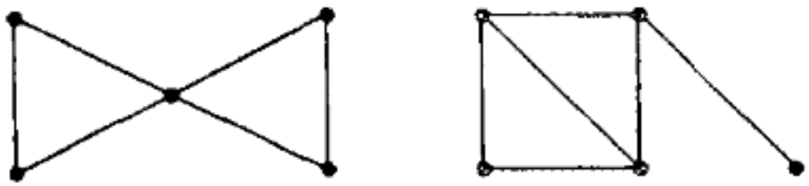

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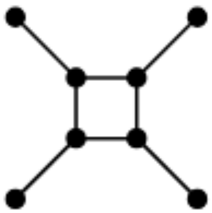
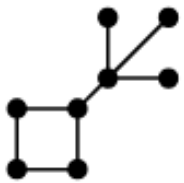
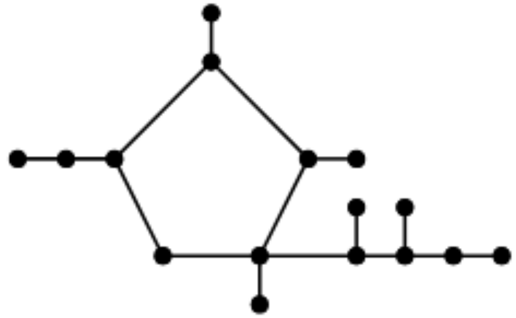
Notice that both of the graphs above also have **triangles** (3-cycles).

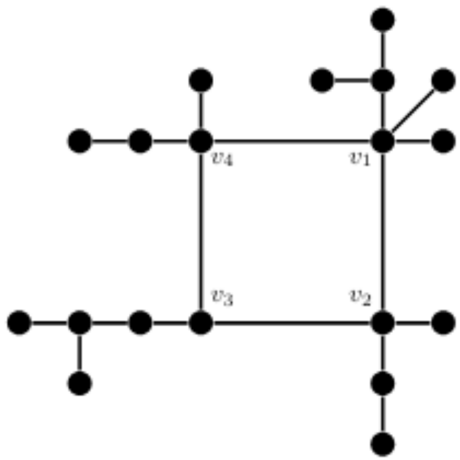
Light



Part 2: Light on the trees







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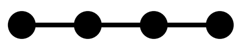
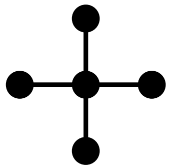
$T =$  $T' =$ 

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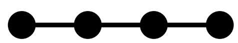
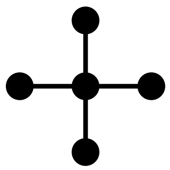
$T =$  $T' =$ 

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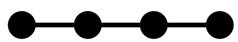
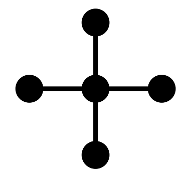
$T =$  $T' =$ 

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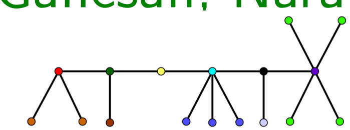

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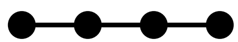
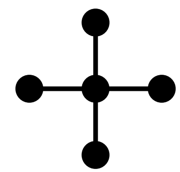
$T =$  $T' =$ 

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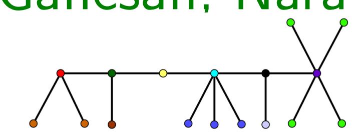
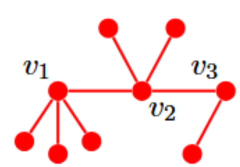


- X_T can reconstruct trees with diameter ≤ 5 ([Aliste-Prieto, de Mier, Orellana, Zamora, 2023],[Gonzalez, Orellana, Tomba, 2024])



Interlude: How many trees?

I will send the winner a postcard, anywhere in the world.

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How many individual trees are there presently on planet Earth, to the nearest ten billion?

(According to the estimation and prognostication of a 2015 Nature article.)

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Closest/fastest wins. Submit your answer in the chat.

You have 30 seconds. Your time starts **now**.

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The CSF and unicyclic graphs

Q: What special properties make trees distinguishable by the CSF?

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For trees, it is known that X_G tells us:

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Moreover, can construct an **infinite family** of such pairs of unicyclic graphs with a triangle.

The trouble with triangles

Orellana, Scott:

- Deg. sequence is not recoverable for unicyclic graphs.
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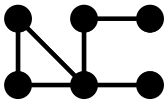
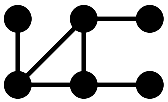
$G =$  $G' =$ 

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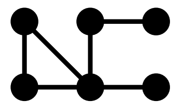
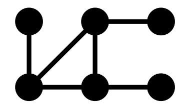
$G =$  $G' =$ 

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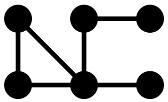
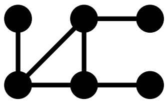
$G =$  $G' =$ 

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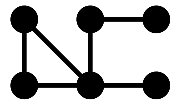
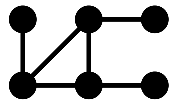
$G =$  $G' =$ 

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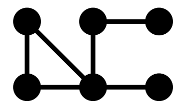
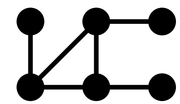
$G =$  $G' =$ 

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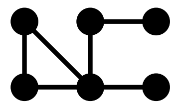
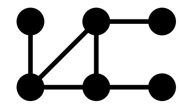
$G =$  $G' =$ 

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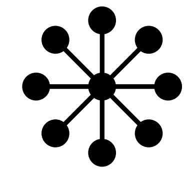
Note: Unicyclic graphs with fixed n and c have the **same chromatic polynomial**! $((k-1)^n + (-1)^c(k-1)^{n-c+1})$

Proposition: For a star graph on $l+1$ vertices, denoted by St_{l+1},

$$X_{St_{l+1}} = \sum_{i=0}^{l} (-1)^i \binom{l}{i} p_{(i+1,1^{l-i})}\,.$$



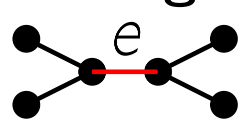
St_9

The star basis:

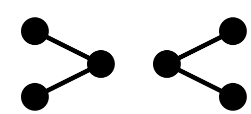
- For any $n \in \mathbb{Z}_{>0}$, set $\mathfrak{st}_n = X_{St_n}$.
- For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, let
  $\mathfrak{st}_\lambda = \mathfrak{st}_{\lambda_1}\mathfrak{st}_{\lambda_2}\cdots\mathfrak{st}_{\lambda_\ell}$.
- $\{\mathfrak{st}_\lambda : \lambda \vdash n\}$ is a basis for the space of symmetric functions of degree n. (Cho, van Willigenburg, 2016)

Operations on the edges of a graph G:

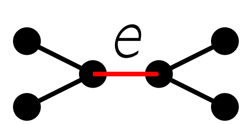
Deletion:



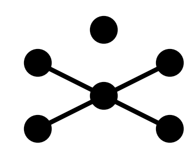
$\xrightarrow{G-e}$



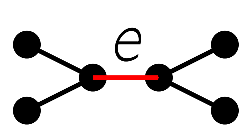
Dot contraction:



$\xrightarrow{(G\odot e)-\ell_e}$



Leaf contraction:



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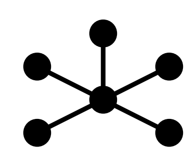
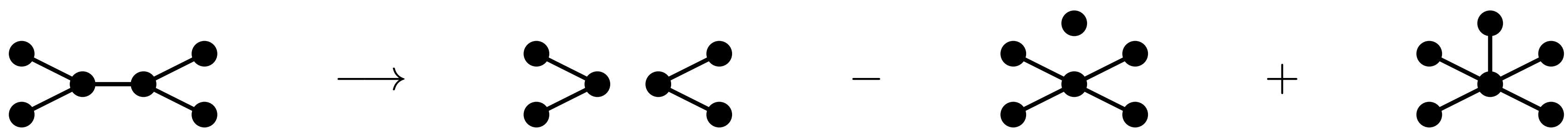


DNC (Deletion-near-Contraction) Relations

Theorem: (Aliste-Prieto, de Mier, Orellana, Zamora, 2023)
For G a graph and e an edge of G,

$$X_G = X_{G-e} - X_{(G\odot e)-\ell_e} + X_{G\odot e}\,.$$

EX:

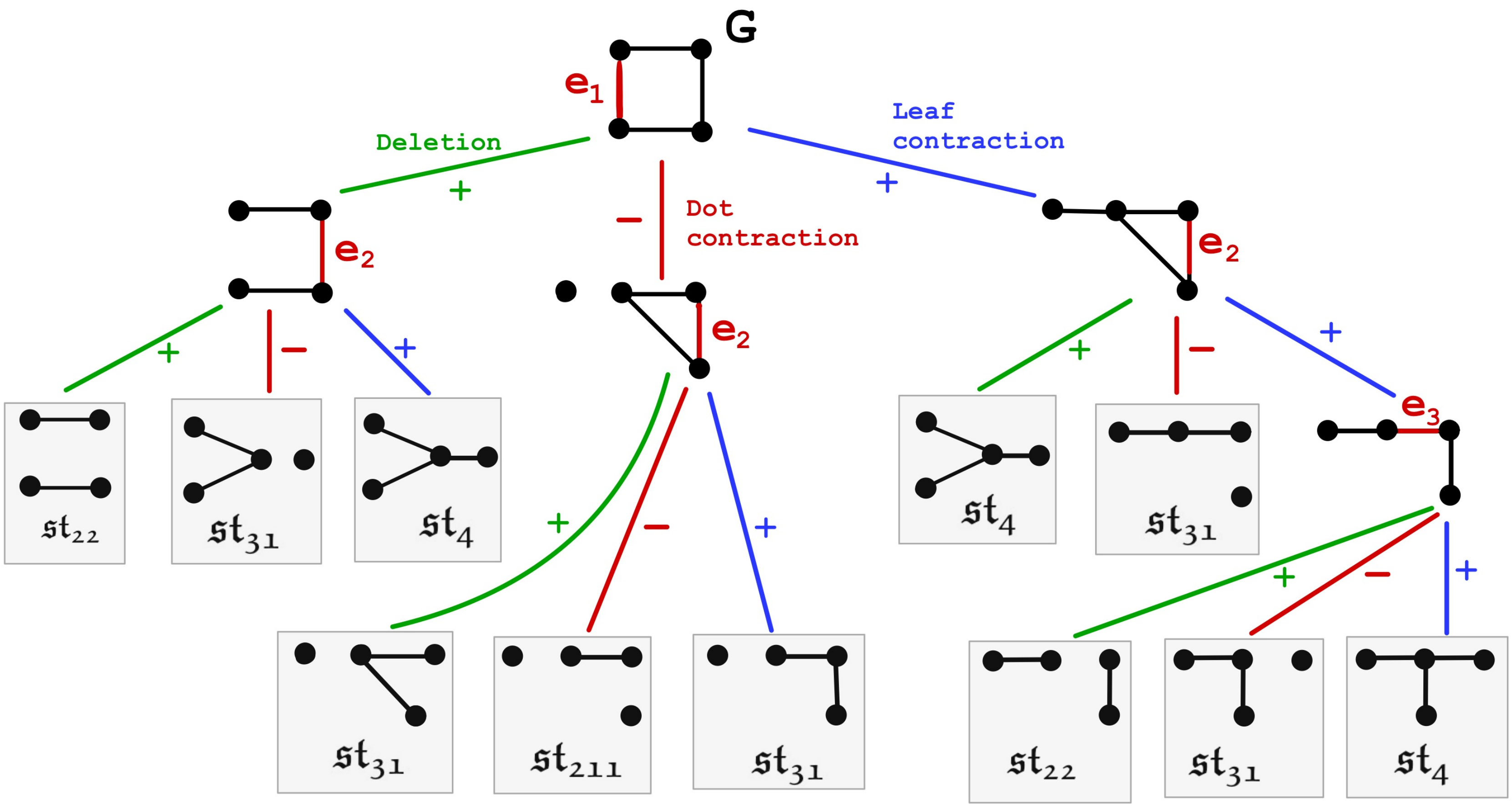


- If e is a leaf edge, then $G - e = (G \odot e) - \ell_e$ and $G \odot e = G$, so

$$X_G = X_{G-e} - X_{(G\odot e)-\ell_e} + X_{G\odot e} = X_G\,.$$

- Stars are the only connected graphs with no internal edges.
- To compute the CSF in the star basis, iteratively apply DNC until no internal edges appear. See also Chmutov–Shah, 2020

Example: star expansion algorithm



$$X_G = 1\mathfrak{st}_{211} + 2\mathfrak{st}_{22} - 5\mathfrak{st}_{31} + 3\mathfrak{st}_4$$

Some applications (B.+JLOPS, 2026)

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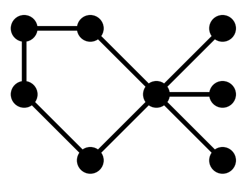

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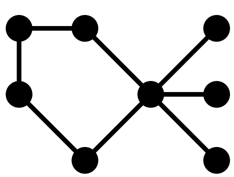

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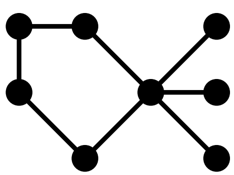

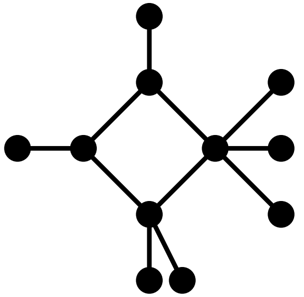

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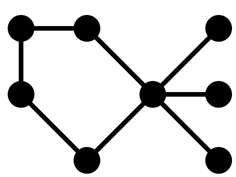

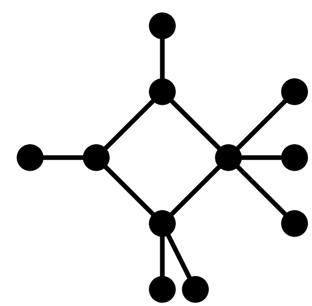

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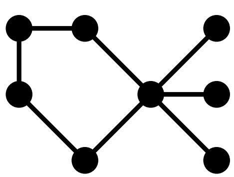

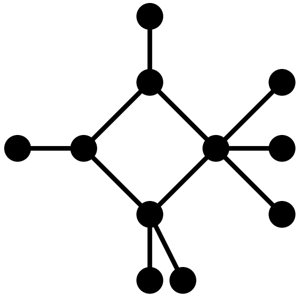

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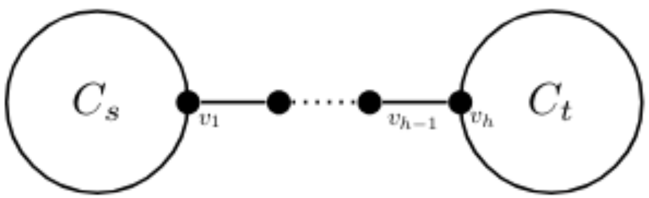
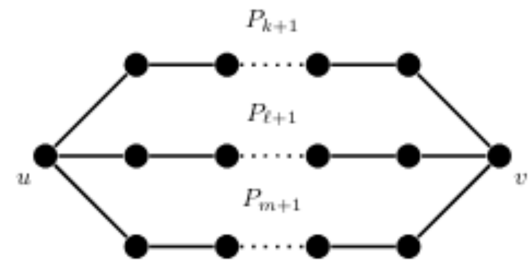
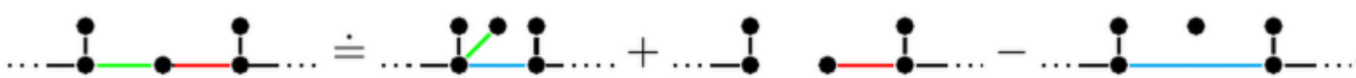
(6) We can recover cycle sizes and the number of edges in common for bicyclic graphs.



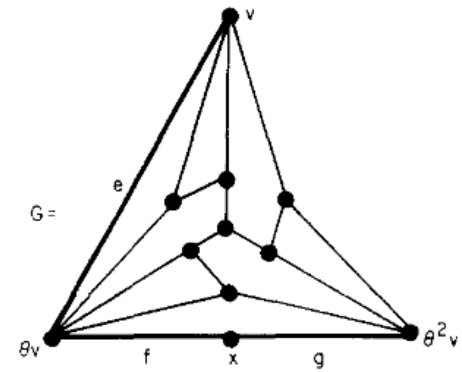


Part 3: Shade on the trees



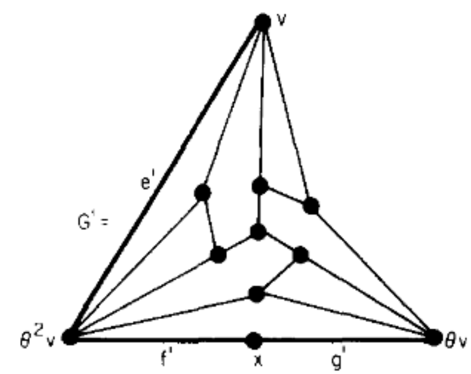
Searching for graphs with the same invariant

Brylawski, 1981: Graphs with the same polychromate.



Sarmiento, 2000: Equivalence of polychromate, U-polynomial (Noble–Welsh), and "bad" CSF.

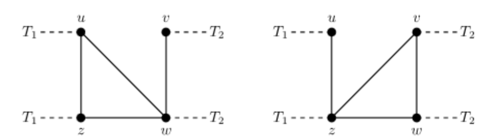
Orellana–Scott, 2014: 3-unicyclic examples with same CSF, modular (“triple") relation. (Also Guay–Paquet, 2013)



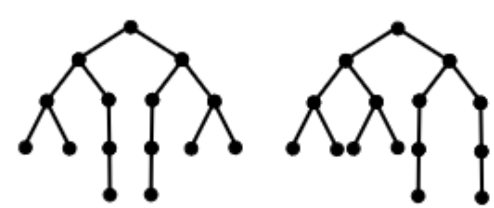
Dahlberg–van Willigenburg, 2017: Generalization of triple relation to c-cycles.

Aliste-Prieto–de Mier–Zamora, 2018: Trees with the same restricted U-polynomial.

Penaguiao, 2020: Proof that the triple relation generates the kernel of $G \mapsto \mathrm{X}_G$.



Aliste-Prieto–Crew–Spirkl–Zamora 2021: Pairs with same (bad) CSF and arbitrary girth



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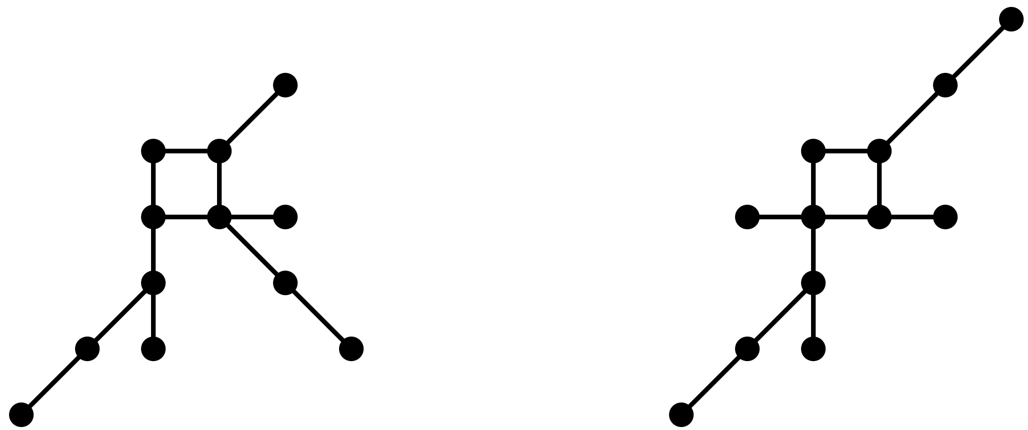

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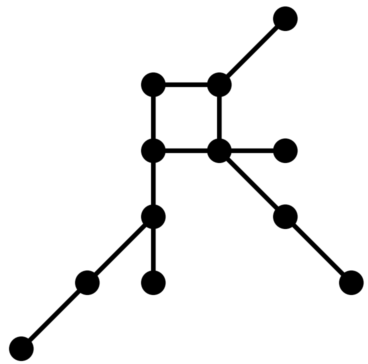

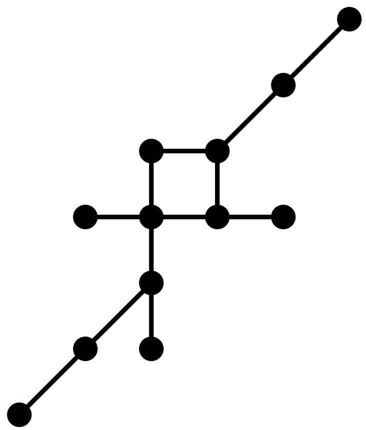

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We found another counterexample on 13 vertices (with a 4-cycle) with the same CSF. These also had the same degree sequence. Also a pair of 16 vertex graphs with a 5-cycle.

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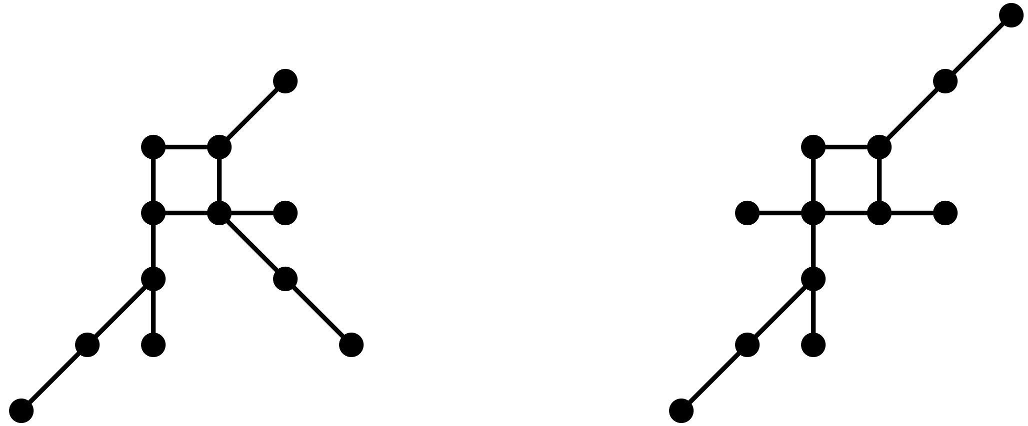
A: (BJLOPS, 2026) No.

Counterexample: Two bipartite graphs on 12 vertices with same CSF:

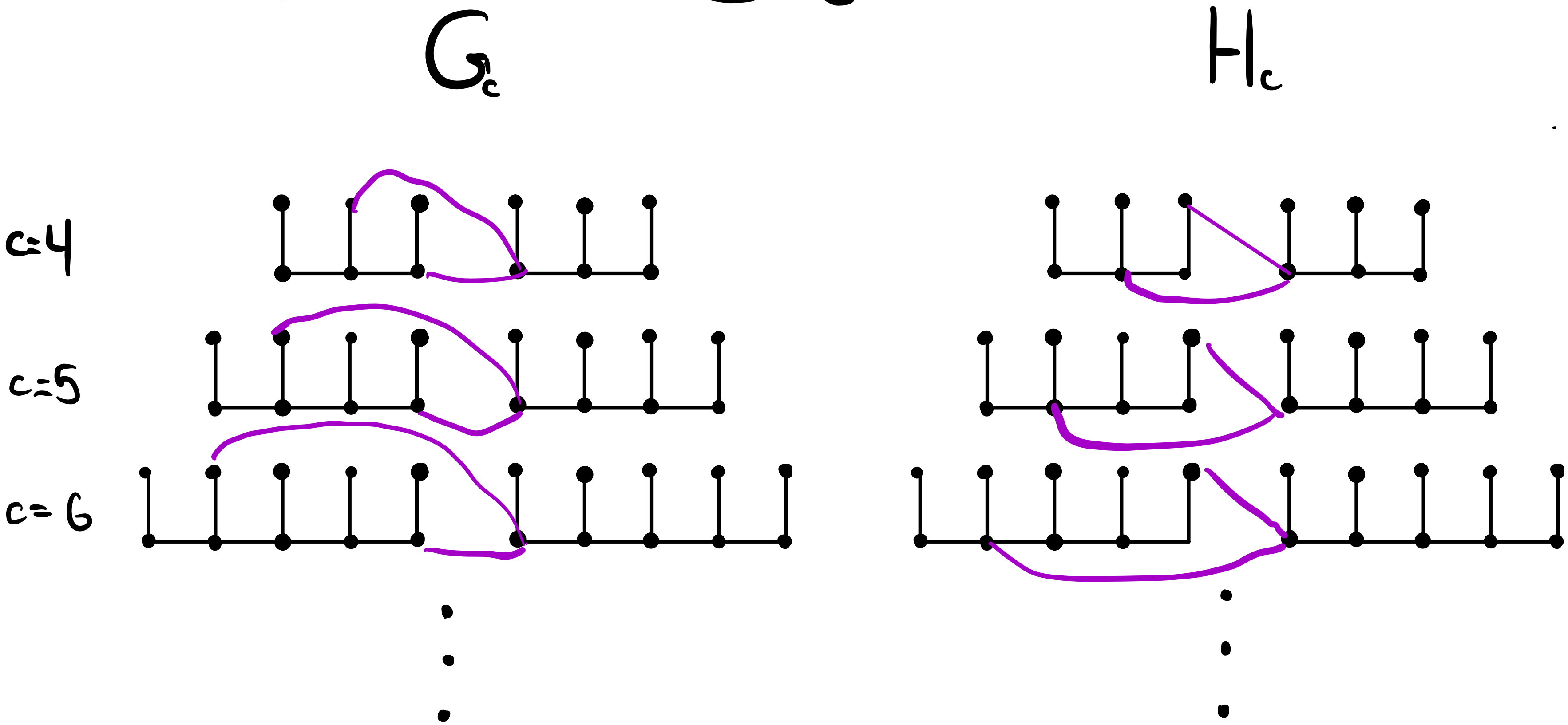


Note: Both have same degree sequence: $(4, 3, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1)$.

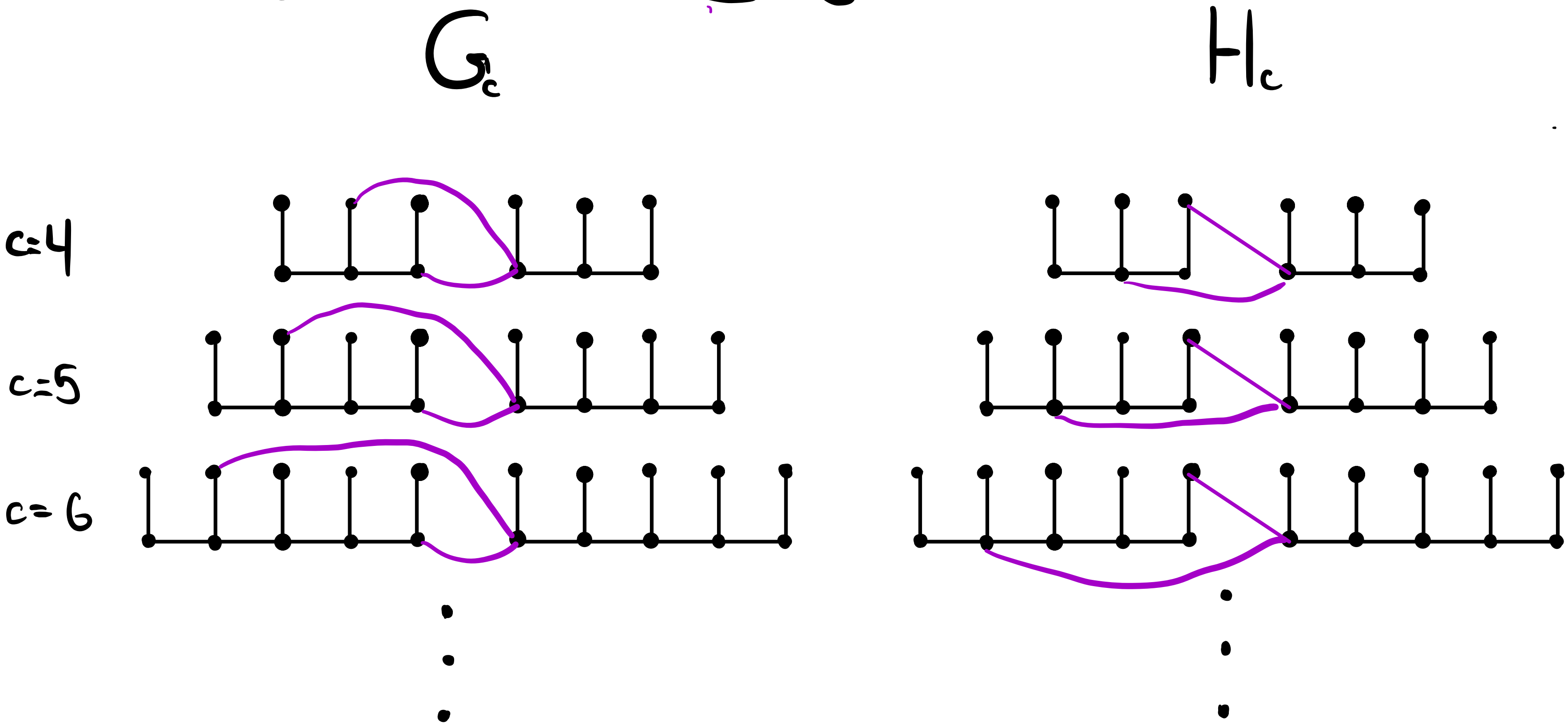
We found another counterexample on 13 vertices (with a 4-cycle) with the same CSF. These also had the same degree sequence. Also a pair of 16 vertex graphs with a 5-cycle.

Q: If two unicyclic graphs share the same CSF where cycle size is > 3, do they share (gen.) deg. seq.? Number of attached trees? Path sequence?

Unicyclic graphs of arbitrary girth with the same CSF



Unicyclic graphs of arbitrary girth with the same CSF



<u>Theorem</u> (B., 26⁺) For each $c \geq 4$, (G_c, H_c) is a nonisomorphic pair with $X_{G_c} = X_{H_c}$

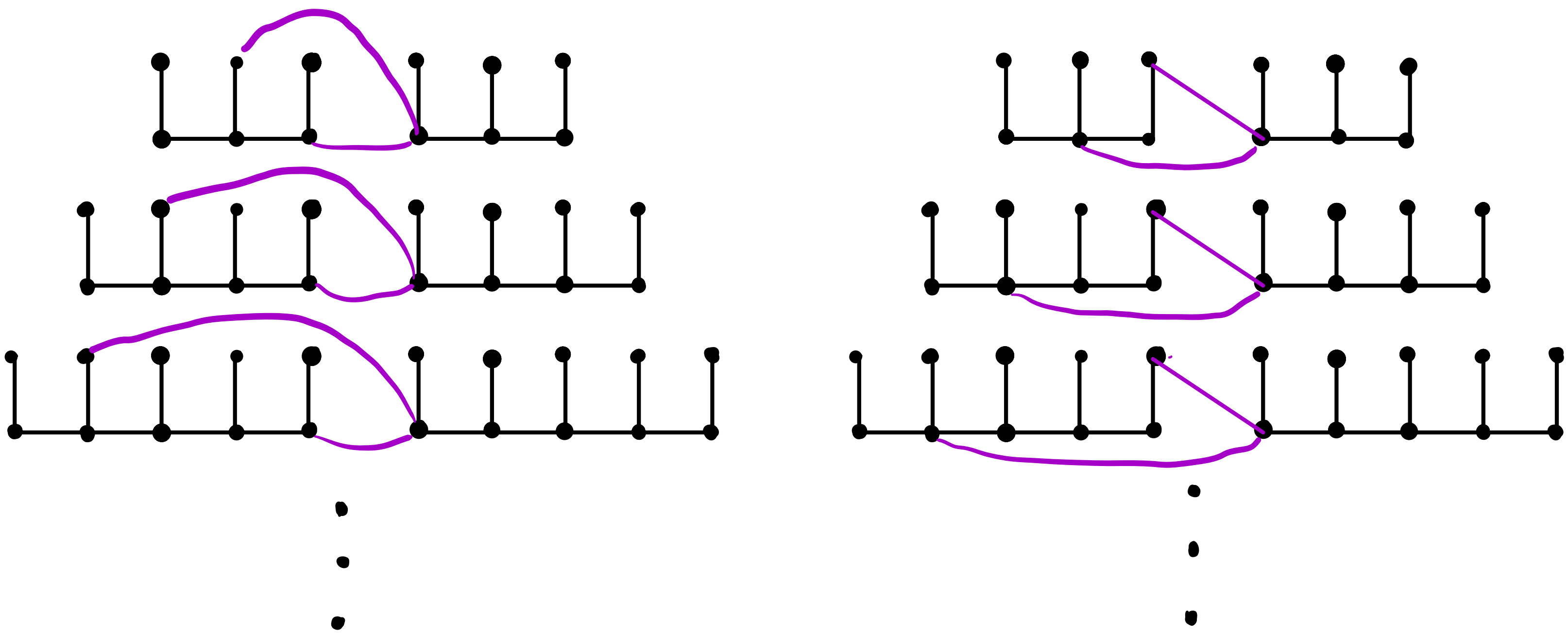
Unicyclic graphs of arbitrary girth with the same CSF

G_c $\qquad\qquad$ H_c

$c=4$

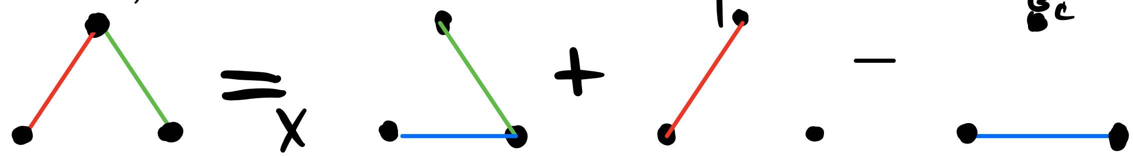
$c=5$

$c=6$



Theorem (B, 26+) For each $c \geq 4$, (G_c, H_c) is a nonisomorphic pair with $X_{G_c} = X_{H_c}$

Proof Use modular relation



repeatedly to write

$$G_c =_x A_c - B_c + T_c - S_c + F_c + \sum_{i=1}^{c-4} (2^i, 3) \sqcup (2^{c-3-i}, 3, 2^{c-2}) - (2^{i+1}) \sqcup (3, 2^{c-4-i}, 3, 2^{c-2}) =_x H_c$$

Thank you!

BJLOPS: “The Chromatic Symmetric Function for Unicyclic Graphs", with Lisa Johnston, Colin Lawson, Rosa Orellana, Jianping Pan, and Chelsea Sato

B: “Unicyclic graphs of arbitrary girth with the same CSF"

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