UNICYCLIC GRAPHS OF ARBITRARY GIRTH WITH THE SAME CHROMATIC SYMMETRIC FUNCTION

ARAM BINGHAM

Abstract. An open question asks whether the chromatic symmetric function (CSF) of a graph distinguishes non-isomorphic trees. While it is known that the CSF does not distinguish unicyclic graphs, examples of pairs of unicyclic graphs with the same CSF and girth larger than 3 were not known until very recently. This manuscript exhibits a sequence of pairs of connected unicyclic graphs of increasing girth which share the same CSF. This also provides the first infinite family of bipartite graphs with the same CSF. Our main technique is to apply a version of the "triple deletion" modular relation, due independently to Guay-Paquet and Orellana–Scott.

1. Introduction

The chromatic symmetric function $\mathbf{X}_G$ of a simple graph G was introduced in [13] by Stanley. For a graph G with vertex set $V = \{v_1, \ldots, v_n\}$, $\mathbf{X}_G$ is defined as

$$\mathbf{X}_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \ldots x_{\kappa(v_n)},$$

where $\kappa : V \to \mathbb{N}$ ranges over all proper colorings of the vertex set.

The "tree isomorphism problem," often attributed to Stanley, asks whether the chromatic symmetric function (CSF) distinguishes non-isomorphic trees. That is, for trees T, T', is it the case that

$$\mathbf{X}_T = \mathbf{X}_{T'} \implies T \cong T'?$$

It has been shown computationally that this is the case for all trees up to 29 vertices [7]. Other results identify specific families of trees which are distinguished by the CSF; see, for instance, [2, 5, 8, 9, 10, 15].

In order to better understand what graphical information the CSF is capable of capturing, some authors have also studied the CSF of unicyclic graphs [10, 11, 4, 14, 3]. It has been known since work of Orellana–Scott [11] that there are infinitely many pairs of non-isomorphic unicyclic graphs containing triangles which have the same CSF. More recently, the author together with Johnston, Lawson, Orellana, Pan and Sato exhibited the first examples of unicyclic graphs of girth 4 and 5 with the same CSF [3]. Based on computational evidence described in that reference, triangle-free examples of non-isomorphic pairs of graphs with the same CSF appear to be much rarer, further substantiating observations made in [1] in which the first triangle-free (but not unicyclic) examples were described.

Here, our sequence of pairs of graphs with the same CSF is based on the smallest pairs of girth 4 and girth 5 unicyclic graphs found in [3]. There is only one other pair of unicyclic graphs with girth at least 4 and with the same CSF on up to 17 vertices. It is possible that this other pair (on 13 vertices and a 4-cycle) is the first in another infinite sequence of increasing girth pairs with the same CSF. However, as the proof of the main result of this manuscript shows, we currently lack a clear understanding of the pattern to which such pairs belong.

Nevertheless, the existence of arbitrary girth pairs of unicyclic graphs with the same CSF offers some insight into the tree isomorphism problem. Our pairs of non-isomorphic unicyclic graphs differ from one another by only two edges. Through repeated application of a modular "triple" relation [6, 11][1], we transform the CSF of each graph in our pair into identical expressions. This *modular relation* has been shown by Penaguião to generate (together with isomorphism relations on graphs) the kernel of a Hopf algebra map

Date: June 9, 2026.
2020 *Mathematics Subject Classification.* Primary 05E05, 05C60; Secondary 05C38.
Key words and phrases. chromatic symmetric functions, unicyclic graphs.

[1] This is actually a four term relation, sometimes also called "triple deletion."

from graphs to symmetric functions, implying that it is sufficient to determine all linear relations between CSFs of non-isomorphic graphs [12].

On the other hand, the lengthy sequence of calculations we have found necessary to show equality of CSFs indicates the subtlety of the triple relation. If examples of pairs of non-isomorphic trees with the same CSF exist they may also differ by a small number of edges without being easily related by the modular relation. Our difficulty in finding the correct application of these relations to prove equality of CSFs reveals the complexity of CSF computation and identity testing. Despite significant experimental work suggesting that the CSF distinguishes trees (see [7] and references therein), we interpret our examples as pointing towards the possiblity that the CSF does not distinguish trees, as they yield the "most tree-like" graphs yet found which the CSF cannot distinguish.

This paper is organized as follows. In Section 2 we recall the triple deletion relation and outline its application. Next, in Section 3 we describe our two sequences of unicyclic graphs which form the pairs with identitcal CSF. Section 4 is devoted to the calculations involved in verifying that members of each sequence of graphs give the same CSF.

Acknowledgments. The author is supported by FONDECYT-ANID grant 3250472.

2. Definitions and preliminaries

All our graphs are simple. We recall here a version of the modular relation of [11, Corollary 3.2]. For a graph $G = (V, E)$ with $v_1, v_2 \in V$ but $e = \{v_1, v_2\}$, we write $G + \{v_1, v_2\}$ or $G + e$ to denote the graph with vertex set V and edge set $E \cup \{\{v_1, v_2\}\}$. Similarly, write $G - \{v_1, v_2\}$ or $G - e$ for the usual edge deletion operation. Now let graph G contain incident edges $e_1 = \{u, v\}$ and $e_2 = \{v, w\}$, with $e_3 = u, w$. Then the modular relation states

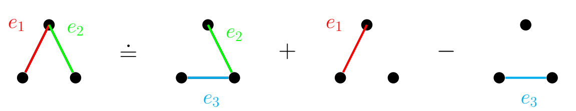
$$\mathbf{X}_G = \mathbf{X}_{G-e_1+e_3} + \mathbf{X}_{G-e_2} - \mathbf{X}_{G-e_1-e_2+e_3}. \tag{2.1}$$

Of course, one obtains a similar relation by interchanging e_1 and e_2.

Since the computations in this article will involve identities only between CSFs of graphs, we will use the symbol $\doteq$ to denote an equality of the CSFs of the indexing graphs. That is, instead of (2.1), we will write equations (with appropriate parentheses) of the form

$$G \doteq (G - e_1 + e_3) + (G - e_2) - (G - e_1 - e_2 + e_3). \tag{2.2}$$

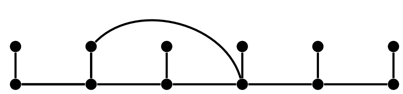
We find it easier to depict and apply this relation visually as follows. Consider the graph G with u, v, w as its only vertices, and e_1, e_2, e_3 as above. Then (2.2) becomes



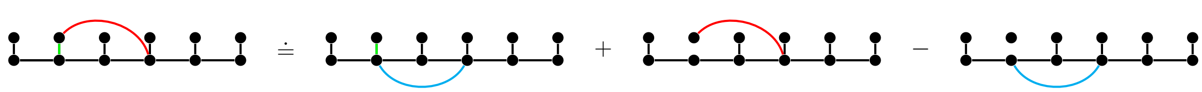
(2.3)

We use this color scheme when helpful to visually depict equations of the form (2.2). This is illustrated in the next example.

Example 2.1. Let G be the unicyclic graph on 12 vertices depicted below.



Identifying two incident edges of the graph as e_1 and e_2, we have the equation below.



(2.4)

As many of the graphs that appear in our computations are *caterpillars* or forests of caterpillars, we will also introduce some notation to represent these concisely. Caterpillars are trees such that the induced subgraph on all non-leaf vertices is just a path. The length ℓ of this path is referred to as the *length* of the caterpillar. This path subgraph is also called the *spine* of the caterpillar.

Non-leaf vertices are called *internal vertices*. For each internal vertex v, the subgraph induced by v together with all its leaf neighbors is called a *leaf component*. Choosing an order for the spine vertices $v_1, \ldots, v_\ell$ from one end to the other and taking α_i as the size of the leaf component of v_i, a caterpillar may be described by an integer composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. This composition is a unique invariant of the graph up to reversal; that is α and $\alpha^* := (\alpha_\ell, \ldots, \alpha_1)$ refer to the same caterpillar. Note that the only conditions on the parts of the composition are that $\alpha_1, \alpha_\ell \geq 2$ and all other $\alpha_i \geq 1$. As an example, the caterpillar which appears as the middle term on the right hand side of (2.4) corresponds to the composition $(2, 1, 2, 3, 2, 2)$. When referring to a caterpillar T, we will sometimes write $T = \alpha$ to mean that T is the caterpillar indexed by composition α.

Forests of caterpillars will frequently appear in our calculations. For compositions α, β, we will write $\alpha \sqcup \beta$ to refer to the (CSF of the) graph which consists of the disjoint union of the two caterpillars indexed by those compositions. To make our notation more concise, we may also write

$$(\alpha_1, \ldots, \overbrace{m, \ldots, m}^{k \text{ times}}, \ldots, \alpha_\ell) = (\alpha_1, \ldots, m^k, \ldots, \alpha_\ell).$$

For example, $(2, 3, 2, 2, 2, 2, 4, 3, 2) = (2, 3, 2^4, 4, 3, 2)$ as compositions and as caterpillars. This should not provoke confusion as we will perform no multiplication or exponentiation operations on the entries of our compositions.

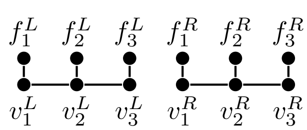
Finally, the special caterpillars of the form (2^k) for some k is called a *comb* graph.

3. Construction

We represent our sequence of pairs of non-isomorphic unicyclic graphs of increasing girth as (G_k, H_k) where k indicates the size of the unique cycle. (G_4, H_4) is depicted below.

Note that these graphs are the same pair that appears in [3, Figure 14]. (G_5, H_5) also appear in that reference as Figure 16.

Our construction of the general pair (G_k, H_k) is as follows. Let $k \geq 4$. Consider first the disjoint union of two comb graphs: $(2^{k-1}) \sqcup (2^{k-1})$. Depict this pair of combs as consisting of a "left" and a "right" comb and label the spine vertices $(v_1^L, \ldots, v_{k-1}^L)$ and $(v_1^R, \ldots, v_{k-1^R})$ respectively. Similarly label the corresponding leaf vertices $(f_1^L, \ldots, f_{k-1}^L)$ and $(f_1^R, \ldots, f_{k-1}^R)$. For example, the graph $(2^3) \sqcup (2^3)$ is depicted as labelled below.

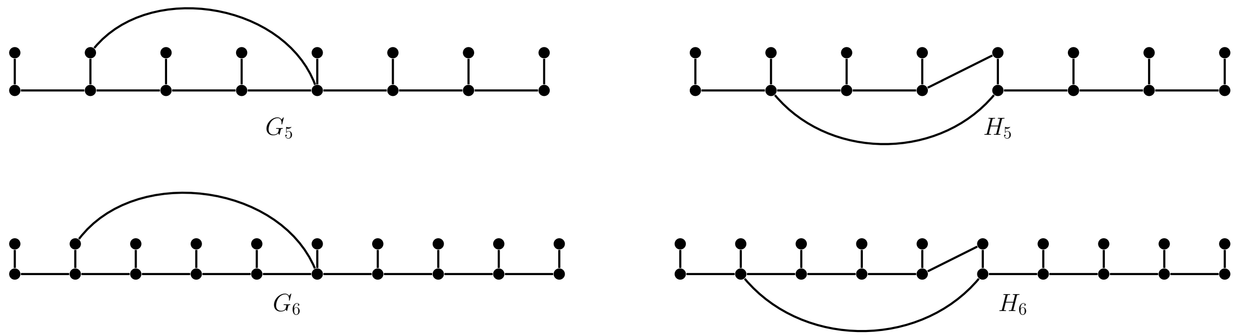


Notice that G_4 can be constructed from $(2^3) \sqcup (2^3)$ by adding edges $\{v_3^L, v_1^R\}$ and $\{f_2^L, v_1^R\}$, while H_4 is constructed by adding edges $\{v_3^L, f_1^R\}$ and $\{v_2^L, v_1^R\}$.

Definition 3.1. For any $k \geq 4$, the pair of graphs (G_k, H_k) are constructed from the union $(2^{k-1}) \sqcup (2^{k-1})$ with vertices as labeled as above, and subject to the following modifications:

- add edges $\{v_{k-1}^L, v_1^R\}$ and $\{f_2^L, v_1^R\}$ to obtain G_k.
- add edges $\{v_{k-1}^L, f_1^R\}$ and $\{v_2^L, v_1^R\}$ to obtain H_k.

We illustrate (G_5, H_5) and (G_6, H_6) below.



Our formula for the CSF of these pairs will consist of (the CSFs of) two other unicyclic graphs and a number of other special caterpillar forests. We now present those graphs.

Definition 3.2. Let A_k and B_k be the unicyclic graphs obtained from the union $(2^{k-1}) \sqcup (2^{k-1})$ with vertices as labeled above subject to the following modifications:

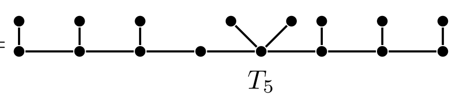
- Add edge $\{v_2^L, v_1^R\}$ and $\{v_k^L, v_1^R\}$ to obtain A_k.
- Starting from A_k, delete edge $\{v_2^L, f_2^L\}$ to obtain B_k.

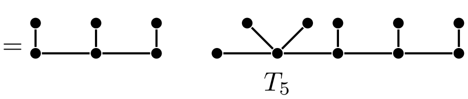
We illustrate A_5 and B_5 below.

Our expression for the CSF of our pairs will involve one other type of caterpillar, and then many graphs which are the disjoint union of two caterpillars. These graphs, like (A_k, B_k), come in pairs with opposite sign in the expression for $\mathbf{X}_{G_k}$. We describe this special caterpillar and the next pair now.

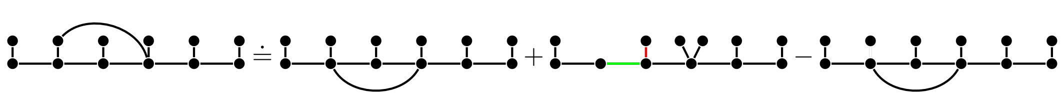
Definition 3.3. Define T_k as the caterpillar on $4k-4$ vertices which corresponds to the composition $(2^{k-2}, 1, 3, 2^{k-2})$. Define S_k as $(2^{k-2}) \sqcup (4, 2^{k-2})$. Finally, define F_k as $(3) \sqcup (2^{k-3}, 3, 2^{k-2})$

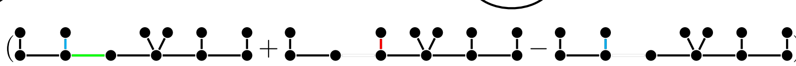
For example, T_5 and S_5 are depicted below.

$T_5 = (2^3, 1, 3, 2^3) =$ 

$S_5 = (2^3) \sqcup (4, 2^3) =$ 

Example 3.4. We illustrate the calculation we will make in general for G_k with G_4. Applying the triple relation (2.2) starting from (2.4), we have



$\doteq A_4 - B_4 + ($  $)$

$\doteq A_4 - B_4 + T_4 - S_4 + F_4.$

In what follows we will also use the notation $\alpha \cdot \beta$ to denote the *concatenation* operation on compositions. That is, if $\alpha = (\alpha_1, \dots, \alpha_l)$ and $\beta = (\beta_1, \dots, \beta_m)$, then $\alpha \cdot \beta = (\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_m)$. We will write $\alpha \odot \beta$ to denote the *near-concatenation* operation on compositions. That is $\alpha \odot \beta = (\alpha_1, \dots, \alpha_{l-1}, \alpha_l + \beta_1, \beta_2, \dots, \beta_m)$.

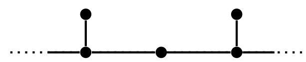
4. Proof

First we present our formula for $\mathbf{X}_{G_k}$. This will be easier to establish since it arises as a cancellation-free computation with straightforward application of the triple deletion relation. We start with a lemma which tells us how to "shuffle" legs from one side of our caterpillars to the other.

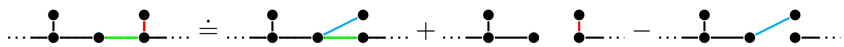
Lemma 4.1. *For a caterpillar with composition of the form $\alpha \cdot (2,1,2) \cdot \beta$ for any α, β, we have*

$$\alpha \cdot (2,1,2) \cdot \beta \doteq \alpha \cdot (2,2,1) \cdot \beta + \alpha \cdot (3) \sqcup (2) \cdot \beta - \alpha \cdot (2,2) \sqcup (1) \odot \beta.$$

Proof. Depict the caterpillar $\alpha \cdot (2,1,2) \cdot \beta$ as



where the dotted lines represent the continuation into the arbitrary compositions α and β on either side. Then, applying (2.2), we have



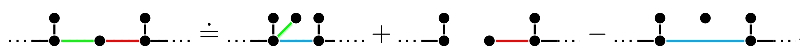
The claimed formula is evident from the above equation. $\square$

We will also need the following later on.

Lemma 4.2. *For a caterpillar with composition of the form $\alpha \cdot (2,1,2) \cdot \beta$ for any α, β, we have*

$$\alpha \cdot (2,1,2) \cdot \beta \doteq \alpha \cdot (3,2) \cdot \beta + \alpha \cdot (2) \sqcup (3) \cdot \beta - (1) \sqcup \alpha \cdot (2,2) \cdot \beta.$$

Proof. This time, apply (2.2) as



$\square$

Lemma 4.3. *Let $k \geq 5$ and consider the caterpillar $(2,1,2^{k-3},3,2^{k-2})$. We have*

$$(2,1,2^{k-3},3,2^{k-2}) \doteq T_k - S_k + F_k + \sum_{i=1}^{k-4}((2^i,3) \sqcup (2^{k-3-i},3,2^{k-2}) - (2^{i+1}) \sqcup (3,2^{k-4-i},3,2^{k-2})).$$

Proof. Applying Lemma 4.1 to the $(2,1,2)$ pattern in the initial caterpillar (α is the empty composition), we have

$$\begin{aligned}(2,1,2^{k-3},3,2^{k-2}) &\doteq (2,2,1,2^{k-4},3,2^{k-2}) + (3) \sqcup (2^{k-3},3,2^{k-2}) - (2,2) \sqcup (3,2^{k-5},3,2^{k-2}) \\ &= (2,2,1,2^{k-4},3,2^{k-2}) - (2,2) \sqcup (3,2^{k-5},3,2^{k-2}) + F_k\end{aligned}$$

Reapplying Lemma 4.1 to the first term on the right-hand side, we have

$$\begin{aligned}(2,1,2^{k-3},3,2^{k-2}) \doteq ((2,2,2,1,2^{k-5},3,2^{k-2}) + (2,3) \sqcup (2^{k-4},3,2^{k-2}) - (2,2,2) \sqcup (1) \odot (2^{k-5},3,2^{k-2})) \\ - (2,2) \sqcup (3,2^{k-5},3,2^{k-2}) + F_k.\end{aligned}$$

If $k = 5$ then this is exactly

$$(2,1,2^2,3,2^3) = T_5 - S_5 + F_5 + \sum_{i=1}^{1}(2^i,3) \sqcup (2^{2-i},3,2^3) - (2^{i+1}) \sqcup (3,3,2^3).$$

Otherwise, continue to apply Lemma 4.1 to the first term on the right-hand side $k-5$ more times to obtain the claimed formula. $\square$

Lemma 4.4. *For $k \geq 4$, the CSF $\mathbf{X}_{G_k}$ can be written as*

$$G_k \doteq A_k - B_k + T_k - S_k + F_k + \sum_{i=1}^{k-4}((2^i,3) \sqcup (2^{k-3-i},3,2^{k-2}) - (2^{i+1}) \sqcup (3,2^{k-4-i},3,2^{k-2})).$$

Proof. The $k = 4$ case is computed in Example 3.4. For $k \geq 5$, as in the example, a first application of (2.2) to the edges $\{f_2^L, v_1^R\}$ and $\{v_2^L, f_2^L\}$ yields

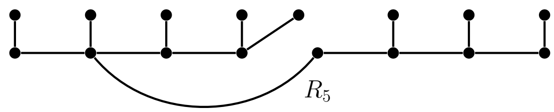
$$G_k \doteq A_k - B_k + (2, 1, 2^{k-3}, 3, 2^{k-2}).$$

Then, applying Lemma 4.3 to the last term on the right-hand side completes the proof. $\square$

Now we will show that $\mathbf{X}_{H_k}$ can be expressed identically. As with $\mathbf{X}_{G_k}$, the A_k and B_k terms are obtained quickly, though the rest will now require more steps and cancellations. We begin by introducing some other graphs that appear in the calculation.

Definition 4.5. Let R_k be the graph on $4k - 4$ vertices obtained from $(2^{k-1}) \sqcup (2^{k-1})$ by adding edges $\{v_2^L, v_1^R\}$ and $\{v_{k-1}^L, f_1^R\}$, and deleting edge $\{v_1^R, f_1^R\}$.

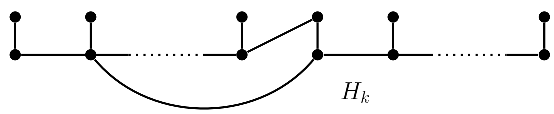
For example, R_5 is depicted below.



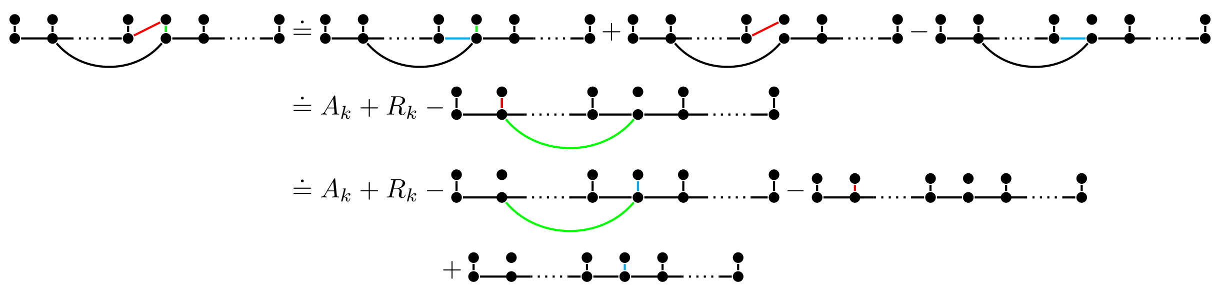
Lemma 4.6. *The CSF $\mathbf{X}_{H_k}$ can be expressed*

$$H_k \doteq A_k - B_k + R_k + (1) \sqcup (2, 1, 2^{2k-4}) - (1) \sqcup (2^{k-2}, 1, 2^{k-1}).$$

Proof. We compute using (2.2), depicting H_k as below.



Hence,



$$\doteq A_k - B_k + R_k + (1) \sqcup (2, 1, 2^{2k-4}) - (1) \sqcup (2^{k-1}, 1, 2^{k-2}),$$

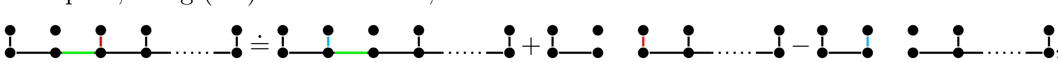
as claimed (after reversing the long composition in the last term). $\square$

Lemma 4.7. *For $k \geq 4$,*

$$(1) \sqcup (2, 1, 2^{2k-4}) - (1) \sqcup (2^{k-2}, 1, 2^{k-1}) \doteq \sum_{i=1}^{k-3} (1) \sqcup (2^{i-1}, 3) \sqcup (2^{2k-3-i}) - (1) \sqcup (2^{i+1}) \sqcup (3, 2^{2k-5-i}).$$

Proof. Depict $(1) \sqcup (2, 1, 2^{2k-4})$ as below.

Then we compute, using (2.2) or Lemma 4.1,



$$(1) \sqcup (2, 1, 2^{2k-4}) \doteq (1) \sqcup (2, 2, 1, 2^{2k-5}) + (1) \sqcup (3) \sqcup (2^{2k-4}) - (1) \sqcup (2, 2) \sqcup (3, 2^{2k-6}).$$

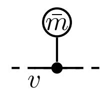
Continue to apply Lemma 4.1 to the $(2, 1, 2)$ pattern of the long composition of the first term on the right-hand side until the 1 is at position $k-1$. This means applying the triple deletion relation $k-3$ times to yield

$$(1) \sqcup (2, 1, 2^{2k-4}) \doteq (1) \sqcup (2^{k-2}, 1, 2^{k-1}) + \sum_{i=1}^{k-3} (1) \sqcup (2^{i-1}, 3) \sqcup (2^{2k-3-i}) - (1) \sqcup (2^{i+1}) \sqcup (3, 2^{2k-5-i}),$$

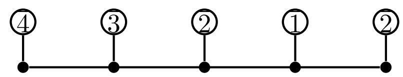
as claimed. ∎

We continue with a lemma which will help us handle R_k. It is similar to Lemma 4.1 in that it lets us move edges along the "spine" of a graph that is nearly a caterpillar. However, we use it to rewrite $\mathbf{X}_{R_k}$ in terms of the CSFs of only forests of caterpillars.

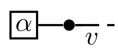
To prepare, first we introduce some graph representation symbology. First, for any number m, let $\bar{m} = m - 1$. If we wish to show that a spine vertex v has $\bar{m}$ leaves attached, we may represent it as below.



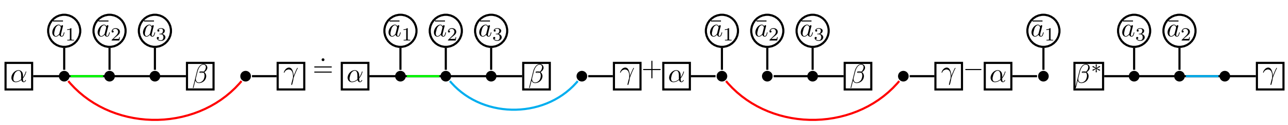
For instance, the caterpillar $(4, 3, 2, 1, 2)$ could be represented as below.



To represent that a vertex v is a neighbor to an extremal spine vertex of a caterpillar $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ (that is, v is a neighbor of the spine vertex of the leaf component of size α_ℓ but is not itself a leaf) we will illustrate this as below.



Lemma 4.8.

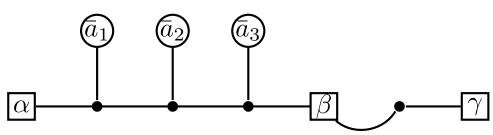


Proof. The proof is indicated by the colored edges of the equation. ∎

Lemma 4.9.

$$R_k \doteq T_k + \sum_{i=1}^{k-3} (2^{k-3-i}, 3) \sqcup (2^{i+1}, 1, 2^{k-2}) - (2^{i+1}) \sqcup (3, 2^{k-3-i}, 1, 2^{k-2}).$$

Proof. This follows by applying Lemma 4.8 first to R_k, and then to each term which is not a forest of caterpillars on the right-hand side until we reach the graph of the form below (in the notation of the lemma).



$= T_k$.

The terms in the summation are straightforwardly recovered as well. ∎

Lemma 4.10.

$$\left(\sum_{i=1}^{k-3} (2^{k-3-i}, 3) \sqcup (2^{i+1}, 1, 2^{k-2}) - (2^{i+1}) \sqcup (3, 2^{k-3-i}, 1, 2^{k-2}) \right) + (1) \sqcup (2, 1, 2^{2k-4}) - (1) \sqcup (2^{k-2}, 1, 2^{k-1})$$

$$\doteq -S_k + F_k + \sum_{i=1}^{k-4} ((2^i, 3) \sqcup (2^{k-3-i}, 3, 2^{k-2}) - (2^{i+1}) \sqcup (3, 2^{k-4-i}, 3, 2^{k-2})).$$

Proof. Apply Lemma 4.2 to the $(2,1,2)$ patterns in the summation:

$$\begin{aligned}(2^{k-3-i},3)\sqcup(2^{i+1},1,2^{k-2}) &- (2^{i+1})\sqcup(3,2^{k-3-i},1,2^{k-2})\\ &\doteq \left((2^{k-3-i},3)\sqcup(2^{i},3,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-4-i},3,2^{k-2})\right)\\ &\quad+\left((2^{k-3-i},3)\sqcup(2^{i+1})\sqcup(3,2^{k-3})-(2^{i+1})\sqcup(3,2^{k-3-i})\sqcup(3,2^{k-3})\right)\\ &\quad-\left((1)\sqcup(2^{k-3-i},3)\sqcup(2^{i+1},2^{k-2})-(1)\sqcup(2^{i+1})\sqcup(3,2^{k-3-i},2^{k-2})\right),\end{aligned}$$

is valid for all $1\le i\le k-3$ with the sole exception that if $i=k-3$, the term in blue becomes $(2^{k-2})\sqcup(4,2^{k-1})=S_k$. Hence,

$$\begin{aligned}&\sum_{i=1}^{k-3}(2^{k-3-i},3)\sqcup(2^{i+1},1,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-3-i},1,2^{k-2})\\ &\qquad\doteq\sum_{i=1}^{k-4}(2^{k-3-i},3)\sqcup(2^{i},3,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-4-i},3,2^{k-2})\\ &\qquad\quad+(3)\sqcup(2^{k-3},3,2^{k-2})-(2^{k-2})\sqcup(4,2^{k-2})\\ &\qquad\quad-\sum_{i=1}^{k-3}\left((1)\sqcup(2^{k-3-i},3)\sqcup(2^{i+1},2^{k-2})-(1)\sqcup(2^{i+1})\sqcup(3,2^{2k-5-i})\right)\end{aligned}$$

Re-indexing the first summation on the last line and applying Lemma 4.7, this is

$$\begin{aligned}&\sum_{i=1}^{k-3}(2^{k-3-i},3)\sqcup(2^{i+1},1,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-3-i},1,2^{k-2})\\ &\qquad-S_k+F_k+\doteq\sum_{i=1}^{k-4}(2^{k-3-i},3)\sqcup(2^{i},3,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-4-i},3,2^{k-2})\\ &\qquad-\left((1)\sqcup(2,1,2^{2k-4})-(1)\sqcup(2^{k-2},1,2^{k-1})\right).\end{aligned}$$

Thus, the left-hand side of the formula of the lemma (after reindexing the first summation) is

$$-S_k+F_k+\sum_{i=1}^{k-4}\left((2^{i},3)\sqcup(2^{k-3-i},3,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-4-i},3,2^{k-2})\right)$$

as claimed. □

We can now prove our formula.

Theorem 4.11. *For any $k\ge 4$, the CSF $\mathbf{X}_{H_k}$ can be written as*

$$H_k\doteq A_k-B_k+T_k-S_k+F_k+\sum_{i=1}^{k-4}((2^{i},3)\sqcup(2^{k-3-i},3,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-4-i},3,2^{k-2})).$$

Hence, $\mathbf{X}_{G_k}=\mathbf{X}_{H_k}$.

Proof. Substituting for R_k using Lemma 4.9 in the expression of Lemma 4.6, we have

$$\begin{aligned}H_k\doteq A_k-B_k+T_k+&\left(\sum_{i=1}^{k-3}(2^{k-3-i},3)\sqcup(2^{i+1},1,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-3-i},1,2^{k-2})\right)\\ &+(1)\sqcup(2,1,2^{2k-4})-(1)\sqcup(2^{k-2},1,2^{k-1}).\end{aligned}$$

Then, Lemma 4.10 applies directly to terms on the right-hand side to give

$$H_k\doteq A_k-B_k+T_k-S_k+F_k+\sum_{i=1}^{k-4}((2^{i},3)\sqcup(2^{k-3-i},3,2^{k-2})-(2^{i+1})\sqcup(3,2^{k-4-i},3,2^{k-2})),$$

as claimed. □

Corollary 4.12. *There exist infinitely many pairs of non-isomorphic bipartite (unicyclic) graphs with the same CSF.*

References

[1] José Aliste-Prieto, Logan Crew, Sophie Spirkl, and José Zamora. "A vertex-weighted Tutte symmetric function, and constructing graphs with equal chromatic symmetric function". English. In: *Electron. J. Comb.* 28.2 (2021). Id/No p2.1, p. 33. ISSN: 1077-8926. DOI: 10.37236/10018.

[2] José Aliste-Prieto, Anna de Mier, and José Zamora. "On the smallest trees with the same restricted U-polynomial and the rooted U-polynomial". In: *Discrete Mathematics* 344.3 (2021), p. 112255.

[3] Aram Bingham, Lisa Johnston, Colin Lawson, Rosa Orellana, Jianping Pan, and Chelsea Sato. *The Chromatic Symmetric Function for Unicyclic Graphs*. Preprint, arXiv:2505.06486 [math.CO] (2025). 2025. URL: https://arxiv.org/abs/2505.06486.

[4] Samantha Dahlberg and Stephanie Van Willigenburg. "Lollipop and Lariat Symmetric Functions". In: *SIAM Journal on Discrete Mathematics* 32 (Feb. 2017).

[5] Arunkumar Ganesan, Narayanan Narayanan, BV Raghavendra Rao, and Sagar S Sawant. "Proper q-caterpillars are distinguished by their Chromatic Symmetric Functions". In: *Discrete Mathematics* 347.11 (2024), p. 114162.

[6] Mathieu Guay-Paquet. *A modular relation for the chromatic symmetric functions of (3+1)-free posets*. Preprint, arXiv:1306.2400 [math.CO] (2013). 2013. URL: https://arxiv.org/abs/1306.2400.

[7] Sam Heil and Caleb Ji. "On an algorithm for comparing the chromatic symmetric functions of trees". In: *Australas. J Comb.* 75 (2018), pp. 210–222.

[8] Jake Huryn and Sergei Chmutov. "A few more trees the chromatic symmetric function can distinguish". In: *Involve: A Journal of Mathematics* 13.1 (2020), pp. 109–116.

[9] Martin Loebl and Jean-Sébastien Sereni. "Isomorphism of weighted trees and Stanley's isomorphism conjecture for caterpillars". In: *Ann. Inst. Henri Poincaré D* 6.3 (2019), pp. 357–384.

[10] Jeremy L Martin, Matthew Morin, and Jennifer D Wagner. "On distinguishing trees by their chromatic symmetric functions". In: *Journal of Combinatorial Theory, Series A* 115.2 (2008), pp. 237–253.

[11] Rosa Orellana and Geoffrey Scott. "Graphs with equal chromatic symmetric functions". In: *Discrete Mathematics* 320 (2014), pp. 1–14.

[12] Raul Penaguiao. "The kernel of chromatic quasisymmetric functions on graphs and hypergraphic polytopes". English. In: *J. Comb. Theory, Ser. A* 175 (2020). Id/No 105258, p. 41. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2020.105258.

[13] R. P. Stanley. "A symmetric function generalization of the chromatic polynomial of a graph". In: *Adv. Math.* 111.1 (1995), pp. 166–194.

[14] David G. L. Wang and Monica M. Y. Wang. "The e-positivity of two classes of cycle-chord graphs". In: *J. Algebraic Comb.* 57.2 (Oct. 2022), pp. 495–514.

[15] Yuzhenni Wang, Xingxing Yu, and Xiao-Dong Zhang. "A class of trees determined by their chromatic symmetric functions". In: *Discrete Mathematics* 347.9 (2024), p. 114096.

(A. Bingham) Departamento de Matemáticas, Universidad de Chile, 3425 Las Palmeras, Ñuñoa, Santiago, CL
Email address: aram@matmor.unam.mx